

# Generating matrices of $C$ -nomial coefficients and their spectra

EMRAH KILIC<sup>a</sup>, PANTELIMON STĂNICĂ<sup>b</sup>

<sup>a</sup>TOBB University of Economics and Technology  
Mathematics Department, 06560 Ankara, Turkey; [ekilic@etu.edu.tr](mailto:ekilic@etu.edu.tr)

<sup>b</sup>Department of Applied Mathematics, Naval Postgraduate School  
Monterey, CA 93943, USA; [pstanica@nps.edu](mailto:pstanica@nps.edu)

April 20, 2010

## Abstract

In this paper, we consider a generalization of binomial coefficients, called  $C$ -nomial coefficients, dependent upon a sequence  $\{u_n\}_n$ , with indices in arithmetic progressions. We obtain a general recurrence relation and a generating matrix, and point out some new relationships between these coefficients and the generalized Pascal matrices. Further, we obtain generating functions, combinatorial representations, and many new interesting identities and properties of these coefficients.

## 1 Introduction

Given that binomial coefficients are very important tools in combinatorics, much effort has been devoted to generalize these objects. We mention here the work of Bachmann [1], Carmichael [3], Fontené [6], Ward [25], as well as the more recent [7, 9, 13, 19, 20, 22, 23].

We define the generalized  $C$ -nomial coefficients formed with the terms of a sequence  $C = \{C_n\}_n$ : for  $n \geq m \geq 1$

$$\begin{bmatrix} n \\ m \end{bmatrix}_C = \frac{C_1 C_2 \dots C_n}{(C_1 C_2 \dots C_{n-m}) (C_1 C_2 \dots C_m)}$$

with  $\begin{bmatrix} n \\ 0 \end{bmatrix}_C = \begin{bmatrix} n \\ n \end{bmatrix}_C = 1$ . One can define the  $n$ th  $C$ -factorial by  $C_n! = C_1 C_2 \dots C_n$ . When  $C_n = F_n$  is the Fibonacci sequence, the numbers  $\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m!}{F_k! F_{m-k}!}$  are called *Fibonomials*, and if  $C_n = (q^n - 1)/(q - 1)$ , the numbers  $\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{(q^m - 1) \dots (q^{m-k+1} - 1)}{(q - 1) \dots (q^k - 1)}$  are called  $q$ -binomial, or *Gaussian coefficients*.

Bachmann [1, p. 81], Carmichael [3, p. 40], and Jarden and Motzkin [13], all showed that if  $u = \{u_n\}_n$  is a Lucas sequence; that is, if  $u_0 = 0$ ,  $u_1 = 1$  and satisfies the recurrence  $u_{n+2} = Au_{n+1} + Bu_n$  for all nonnegative integers  $n$  with some nonzero integers  $A$  and  $B$  such that the quadratic equation  $x^2 - Ax - B = 0$  has two distinct roots  $\alpha$  and  $\beta$  whose

Report Documentation Page			Form Approved OMB No. 0704-0188		
Public reporting burden for the collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington VA 22202-4302. Respondents should be aware that notwithstanding any other provision of law, no person shall be subject to a penalty for failing to comply with a collection of information if it does not display a currently valid OMB control number.					
1. REPORT DATE <b>20 APR 2010</b>		2. REPORT TYPE		3. DATES COVERED <b>00-00-2010 to 00-00-2010</b>	
4. TITLE AND SUBTITLE <b>Generating matrices of C-nomial coefficients and their spectra</b>			5a. CONTRACT NUMBER		
			5b. GRANT NUMBER		
			5c. PROGRAM ELEMENT NUMBER		
6. AUTHOR(S)			5d. PROJECT NUMBER		
			5e. TASK NUMBER		
			5f. WORK UNIT NUMBER		
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) <b>Naval Postgraduate School, Department of Applied Mathematics, Monterey, CA, 93943</b>			8. PERFORMING ORGANIZATION REPORT NUMBER		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)			10. SPONSOR/MONITOR'S ACRONYM(S)		
			11. SPONSOR/MONITOR'S REPORT NUMBER(S)		
12. DISTRIBUTION/AVAILABILITY STATEMENT <b>Approved for public release; distribution unlimited</b>					
13. SUPPLEMENTARY NOTES <b>accepted Proc. International Conf. Fibonacci Numbers &amp; Applic. 2010.</b>					
14. ABSTRACT <b>In this paper, we consider a generalization of binomial coefficients, called C{nomial coefficients, dependent upon a sequence function, with indices in arithmetic progressions. We obtain a general recurrence relation and a generating matrix, and point out some new relationships between these coefficients and the generalized Pascal matrices. Further, we obtain generating functions, combinatorial representations, and many new interesting identities and properties of these coefficients.</b>					
15. SUBJECT TERMS					
16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT <b>Same as Report (SAR)</b>	18. NUMBER OF PAGES <b>13</b>	19a. NAME OF RESPONSIBLE PERSON
a. REPORT <b>unclassified</b>	b. ABSTRACT <b>unclassified</b>	c. THIS PAGE <b>unclassified</b>			

ratio is not a root of unity, then all the  $C$ -nomial coefficients are integers (by abuse of notation, we will call all such coefficients  $C$ -nomials, if the sequence is not special, but understood from context). The Fibonomial coefficients are particular cases of this instance with  $A = B = 1$ . When  $A = q + 1$  and  $B = -q$ , where  $q > 1$  is some fixed integer, the  $C$ -nomial coefficients become the  $q$ -binomial coefficients.

The Fibonomial coefficients satisfy the relation

$$\begin{bmatrix} n \\ m \end{bmatrix}_F = F_{m+1} \begin{bmatrix} n-1 \\ m \end{bmatrix}_F + F_{n-m-1} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_F.$$

We now take the case of a Lucas sequences with  $A \in \mathbb{Z}, B = 1$ , say  $u_n$  and its companion  $v_n$

$$\begin{aligned} u_n &= Au_{n-1} + u_{n-2} \\ v_n &= Av_{n-1} + v_{n-2} \end{aligned} \tag{1}$$

where  $u_0 = 0, u_1 = 1$  and  $v_0 = 2, v_1 = A$ , respectively, for all  $n \geq 2$ .

There are some relationships between the Fibonomial coefficients (for more details see [2, 4, 10, 21, 17]), and the generalized Pascal matrix  $P_n$ , which is the  $n \times n$  right-justified matrix whose  $(i, j)$  entry is given by

$$(P_n)_{ij} = \binom{j-1}{j+i-n-1} A^{i+j-n-1}.$$

In [9], Hoggatt considers the  $C$ -nomial coefficients with indices in an arithmetic progression. For  $A$  fixed, he defined the numbers (for  $m \geq n$ )

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{u,k} = \frac{u_k u_{2k} \dots u_{kn}}{(u_k u_{2k} \dots u_{k(n-m)}) (u_k u_{2k} \dots u_{km})},$$

which we call  $k : C$ -nomial coefficients (or generalized  $C$ -nomial coefficients, if  $k$  is not specified). It is straightforward to show that they satisfy the following recurrences:

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{u,k} = u_{km+1} \left\{ \begin{matrix} n-1 \\ m \end{matrix} \right\}_{u,k} + u_{k(n-m)-1} \left\{ \begin{matrix} n-1 \\ m-1 \end{matrix} \right\}_{u,k}$$

and

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{u,k} = u_{km-1} \left\{ \begin{matrix} n-1 \\ m \end{matrix} \right\}_{u,k} + u_{k(n-m)+1} \left\{ \begin{matrix} n-1 \\ m-1 \end{matrix} \right\}_{u,k}.$$

When the sequence  $\{u_n\}_n$  is understood from the context, we will write  $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_k$  instead of  $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{u,k}$ . When  $k = 1$ , the coefficient  $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_{u,1}$  is reduced to  $\left[ \begin{matrix} n \\ m \end{matrix} \right]_u$ .

Let  $\alpha, \beta$  be the roots of the associated equation  $x^2 - Ax - 1 = 0$ . As a special case of [17], we have that for  $r \geq 1$  and  $n > 1$ ,

$$\begin{aligned} u_{rn} &= v_r u_{r(n-1)} + (-1)^{r+1} u_{r(n-2)} \\ v_{rn} &= v_r v_{r(n-1)} + (-1)^{r+1} v_{r(n-2)}. \end{aligned} \tag{2}$$

Using the sequence  $\{v_r\}$ , the authors of [17] define the  $n \times n$  generalized Pascal matrix  $P_n(v_r)$  as follows:

$$P_n(v_r) = \left( v_r^{i+j-n-1} (-1)^{(r+1)(n-j)} \binom{i-1}{n-j} \right)_{1 \leq i, j \leq n}$$

and then show that for all  $m > 0$ , the trace

$$\text{tr}(P_n^m(v_r)) = \frac{u_{rnm}}{u_r}.$$

Observe that  $P_n(v_1) = P_n$ . As an example,

$$P_6(v_r) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & (-1)^{r+1} & v_r \\ 0 & 0 & 0 & 1 & 2(-1)^{r+1}v_r & v_r^2 \\ 0 & 0 & (-1)^{r+1} & 3v_r & 3(-1)^{r+1}v_r^2 & v_r^3 \\ 0 & 1 & 4(-1)^{r+1}v_r & 6v_r^2 & 4(-1)^{r+1}v_r^3 & v_r^4 \\ (-1)^{r+1} & 5v_r & 10(-1)^{r+1}v_r^2 & 10v_r^3 & 5(-1)^{r+1}v_r^4 & v_r^5 \end{pmatrix}.$$

Moreover, they derived that the eigenvalues and the characteristic polynomial of  $P_{n+1}(v_r)$  are

$$\alpha^{rn}, \alpha^{r(n-1)}\beta^r, \dots, \alpha^r\beta^{r(n-1)}, \beta^{rn}. \quad (3)$$

and

$$P_{r,n}(x) = \prod_{j=0}^{n-1} (x - \alpha^{jr}\beta^{r(n-j-1)r}) = \sum_{i=0}^n (-1)^i (-1)^{ri(i-1)/2} \left\{ \begin{matrix} n \\ i \end{matrix} \right\}_r x^{n-i} \quad (4)$$

where the coefficient  $\left\{ \begin{matrix} n \\ i \end{matrix} \right\}_r$  is our generalized  $r : C$ -nomial coefficients.

In [18], we defined the recursive analogue of the entries of the Lehmer matrix and then formulated its determinant via the Fibonomial factorials (or  $F$ -factorials).

In this paper, our purpose is to find generating matrices for generalized  $r : C$ -nomial coefficients  $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_r$  and derive a linear recurrence relation for the generalized  $r : C$ -nomial coefficients. We obtain new general identities, generating functions, combinatorial representations for them and their sums by matrix methods.

## 2 Generalized $C$ -nomial coefficients

In this section, we fix a Lucas sequence  $u_n$  as in (1) and derive a recurrence relation and generating matrix for the generalized  $C$ -nomial coefficients. The case  $k = 2$  and  $r = 1$ , was investigated in [14, 15].

First, we define the sign function

$$s(i, r) = \begin{cases} (-1)^{(i-1)(i-2)/2} & \text{if } r \text{ is odd,} \\ (-1)^{i+1} & \text{if } r \text{ is even,} \end{cases} \quad (5)$$

and extend the generalized  $C$ -nomial coefficients by

$$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}_r = \begin{cases} 0 & \text{if } m > n \text{ and } n \geq 0, \\ s(m, r) & \text{if } m > n \text{ and } n < 0, \\ \frac{u_r u_{2r} \dots u_{nr}}{(u_r u_{2r} \dots u_{mr})(u_r u_{2r} \dots u_{(n-m)r})} & m \leq n. \end{cases}$$

For the sake of compactness, we shall use the following notations, for fixed  $k$  with  $1 \leq i \leq k+1$ :

$$a_{n,i} = s(i, r) \left\{ \begin{matrix} n+k \\ k-i+1 \end{matrix} \right\}_r \left\{ \begin{matrix} n+i-2 \\ i-1 \end{matrix} \right\}_r. \quad (6)$$

As we will be using it later, recall the following identity from [24, p. 176]

$$F_{n+m} = F_{m-1}F_n + F_mF_{n+1},$$

which generalizes (for the sequence  $\{u_n\}$ ), for any integers  $k, m, n$  and positive integer  $r > 0$

$$u_{rk}u_{r(n+m)} = u_{rm}u_{r(n+k)} + (-1)^{rm} u_{rn}u_{r(k-m)}. \quad (7)$$

Now we give the following result.

**Lemma 1.** For  $n > 0$ ,  $1 \leq i \leq k$  and for odd  $r$ ,

$$a_{1,i}a_{n,1} + (-1)^{i-1} a_{n,i+1} = a_{n+1,i},$$

and for even  $r$ ,

$$a_{1,i}a_{n,1} - a_{n,i+1} = a_{n+1,i},$$

where  $a_{n,i}$  be as before.

*Proof.* The proof for  $r$  odd can be found in [14]. Assume  $r$  is even. We simplify the equality  $a_{1,i}a_{n,1} - a_{n,i+1} = a_{n+1,i}$ , reducing it to

$$u_{r(k+1)}u_{r(n+i)} + u_{rn}u_{r(k-i+1)} = u_{ri}u_{r(n+k+1)}.$$

By taking  $k \rightarrow k+1$  in (7), we obtain the second claim of our lemma.  $\square$

For  $k, r \geq 1$ , we define the  $(k+1) \times (k+1)$  companion matrix  $G_{r,k}$  and the matrix  $H_{n,r,k}$  as follows:

$$G_{r,k} = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k+1} \\ 1 & & & \\ & \ddots & & \\ 0 & & 1 & 0 \end{bmatrix} \quad \text{and} \quad (8)$$

$$H_{n,r,k} = \begin{bmatrix} a_{n,1} & a_{n,2} & \dots & a_{n,k+1} \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-k,1} & a_{n-k,2} & \dots & a_{n-k,k+1} \end{bmatrix}.$$

We say that the matrix  $G_{r,k}$  is the *generalized  $C$ -nomial matrix* (with indices in the arithmetic progression  $\equiv 0 \pmod{k}$ ).

Now we give our main result of this section.

**Theorem 2.** For all  $n, r > 0$ ,

$$G_{r,k}^n = H_{n,r,k}.$$

*Proof.* We use induction on  $n$ . The case of  $n = 1$  follows from the definitions of the matrix  $H_{n,r,k}$  and  $k : C$ -nomial coefficients. Suppose that the equation holds for  $n \geq 1$ . Now we show that the equation holds for  $n + 1$ . First, we write

$$G_{r,k}^{n+1} = G_{r,k} G_{r,k}^n = G_{r,k} H_{n,r,k}.$$

From Lemma 1 and matrix multiplication, we get

$$G_{r,k}^{n+1} = G_{r,k} H_{n,r,k} = H_{n+1,r,k},$$

and the theorem is shown.  $\square$

It is valuable to note that when  $k = 1$ , we obtain the following fact (see [16]):

$$G_{1,1} = \begin{bmatrix} v_r & (-1)^{r+1} \\ 1 & 0 \end{bmatrix} \text{ and } H_{n,1,1} = \begin{bmatrix} u_{r(n+1)} & u_{rn} \\ u_{rn} & u_{r(n-1)} \end{bmatrix}.$$

When  $A = r = 1$ , we get the following well known fact:

$$G_{1,1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } H_{n,1,1} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}.$$

When  $A = 1, r = 3, k = 2$ , we get the matrix  $G_{3,2}$  and its  $n$ th power as

$$G_{3,2} = \begin{bmatrix} \frac{F_9}{F_3} & \frac{F_9}{F_3} & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } H_{n,3,2} = \begin{bmatrix} \frac{F_{3(n+1)}F_{3(n+2)}}{F_3F_6} & \frac{F_{3n}F_{3(n+2)}}{F_3F_6} & -\frac{F_{3n}F_{3(n+1)}}{F_3F_6} \\ \frac{F_{3n}F_{3(n+1)}}{F_3F_6} & \frac{F_{3(n-1)}F_{3(n+1)}}{F_3F_6} & -\frac{F_{3(n-1)}F_{3n}}{F_3F_6} \\ \frac{F_{3(n-1)}F_{3n}}{F_3F_6} & \frac{F_{3(n-2)}F_{3n}}{F_3F_6} & -\frac{F_{3(n-2)}F_{3(n-1)}}{F_3F_6} \end{bmatrix}$$

The case  $A = 1, r = 1$  and  $k = 2$  can be found in [15].

From the companion matrix  $G_{r,k}$ , we give a linear recurrence relation for the  $C$ -nomial coefficients.

**Corollary 3.** For  $n, k > 0$ , the  $C$ -nomial coefficients satisfy the following order- $(k + 1)$  linear recursion

$$a_{n+1,1} = \sum_{i=1}^{k+1} a_{1,i} a_{n-i+1,1}$$

or, equivalently,

$$\begin{aligned} \left\{ \begin{matrix} n+k+1 \\ k \end{matrix} \right\}_r &= s(1, r) \left\{ \begin{matrix} k+1 \\ k \end{matrix} \right\}_r \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\}_r + s(2, r) \left\{ \begin{matrix} k+1 \\ k-1 \end{matrix} \right\}_r \left\{ \begin{matrix} n+k-1 \\ k \end{matrix} \right\}_r \\ &\quad + \cdots + s(k, r) \left\{ \begin{matrix} k+1 \\ 1 \end{matrix} \right\}_r \left\{ \begin{matrix} n+1 \\ k \end{matrix} \right\}_r + s(k+1, r) \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_r. \end{aligned}$$

*Proof.* Since  $a_{n,1} = \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\}_r$  and using matrix multiplication, by equating the  $(1, 1)$  entries in the equation  $H_{1,r,k} H_{n,r,k} = H_{n+1,r,k}$ , the claim follows.  $\square$

All identities in the next corollary can be obtained from a property of the matrix multiplication in  $H_{n+1,r,k} = H_{n,r,k}H_{1,r,k}$ ,  $H_{n+m,r,k} = H_{n,r,k}H_{m,r,k}$  and  $H_{n+t,r,k} = H_{n+m,r,k}H_{t-m,r,k}$  for  $n, m > 0$  and  $t > q$ .

**Corollary 4.** *For  $n, r > 0$ , the following identities hold:*

$$\begin{aligned} a_{n-1,1} &= a_{n,k+1}, \\ a_{m+n+1-i,j} &= \sum_{t=1}^{k+1} a_{n+1-i,t} a_{m+1-t,j} \text{ for all } m > 0, \\ a_{n+t+1-i,j} &= \sum_{m=1}^{k+1} a_{n+q+1-i,m} a_{t-q+1-i,j} \text{ for } t > 0 \text{ and } t > q, \\ a_{n+1,1} &= a_{1,1}a_{n,1} + a_{n,2}, \\ a_{n+1,k+1} &= a_{n,1}a_{1,k+1}, \\ a_{n+1,i} &= a_{1,i}a_{n,1} + a_{n,i+1} \text{ for } 2 \leq i \leq k. \end{aligned}$$

### 3 The eigenvalues of the matrix $G_{r,k}$

In this section we determine the eigenvalues of the matrix  $G_{r,k}$ . Since  $G_{r,k}$  is a companion matrix, its characteristic polynomial,  $f_{r,k}(x)$ , can be easily derived.

Thus we have the following result, whose proof is immediate.

**Proposition 5.** *For  $n, k, r > 0$ ,*

$$f_{r,k}(x) = \sum_{t=0}^{k+1} (-s(t, r)) \left\{ \begin{matrix} k+1 \\ t \end{matrix} \right\}_r x^{k+1-t}$$

where the sign function  $s(t, r)$  is as in (5).

Here clearly

$$-s(t, r) = \begin{cases} (-1)^{i(i+1)/2} & \text{if } r \text{ is odd,} \\ (-1)^i & \text{if } r \text{ is even,} \end{cases}$$

and for odd  $r > 0$  and  $n, k > 0$ ,

$$f_{r,k}(x) = \sum_{t=0}^{k+1} (-1)^{i(i+1)/2} \left\{ \begin{matrix} k+1 \\ t \end{matrix} \right\}_r x^{k+1-t}$$

and even  $r > 0$ ,

$$f_{r,k}(x) = \sum_{t=0}^{k+1} (-1)^i \left\{ \begin{matrix} k+1 \\ t \end{matrix} \right\}_r x^{k+1-t}.$$

According to [5, 8, 9, 12], the  $n$ th powers of Fibonacci numbers satisfy the following auxiliary polynomial

$$C_n(x) = \sum_{i=0}^n (-1)^{i(i+1)/2} \left\{ \begin{matrix} n \\ i \end{matrix} \right\}_{F,1} x^{n-i}.$$

In [2, 5], the authors show that the characteristic polynomial of the Pascal matrix  $H_n(p)$  is also equal to the polynomial  $C_n(x) = f_{1,k}(x)$  and so  $C_n(x) = h_{1,n}(x) = f_{1,k}(x)$ .

Consequently, we have the following result.

**Proposition 6.** Let  $\alpha, \beta = \left(p \pm \sqrt{p^2 + 4}\right) / 2$ . The characteristic roots of  $C_{m+1}(x) = f_{1,m}(x)$  are:

$$\begin{cases} \left\{ (-1)^j \alpha^{m-2j}, (-1)^j \beta^{m-2j} \right\}_{j=0,1,\dots,k-1} & \text{if } m = 2k - 1, \\ \left\{ (-1)^k, (-1)^j \alpha^{m-2j}, (-1)^j \beta^{m-2j} \right\}_{j=0,1,\dots,k-1} & \text{if } m = 2k. \end{cases}$$

As a general case of Proposition 6, in [17] we considered the generalized Pascal matrix  $P_n(V_r)$  and then gave its characteristic polynomial  $h_{r,n}(x)$  as in (3). It is clear that the characteristic polynomials of the generalized  $k : C$ -nomial matrix  $G_{r,k}$  and the generalized Pascal matrix  $P_{k+1}(v_r)$  are the same. Also we know that the all roots of the polynomial from [17], that is, the eigenvalues (clearly, distinct) of the matrix  $G_{r,k}$  are

$$\alpha^{rk}, \alpha^{r(k-1)} \beta^r, \dots, \alpha^r \beta^{r(k-1)}, \beta^{rk}.$$

Alternatively, we may give the following proposition.

**Proposition 7.** Let  $\alpha, \beta = \left(p \pm \sqrt{p^2 + 4}\right) / 2$ . The eigenvalues of  $G_{r,k}$  are:

$$\begin{cases} \left\{ (-1)^{jr} \alpha^{r(k-2j)}, (-1)^{jr} \beta^{r(k-2j)} \right\}_{j=0,1,\dots,t-1} & \text{if } k = 2t - 1, \\ \left\{ (-1)^{tr}, (-1)^{jr} \alpha^{r(k-2j)}, (-1)^{jr} \beta^{r(k-2j)} \right\}_{j=0,1,\dots,t-1} & \text{if } k = 2t. \end{cases}$$

As an example, when  $k = r = 4$ , we have

$$G_{4,4} = \begin{bmatrix} a_1 & -b_1 & c_1 & -d_1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } H_{n,4,4} = \begin{bmatrix} a_n & -b_n & c_n & -d_n & a_{n-1} \\ a_{n-1} & -b_{n-1} & c_{n-1} & -d_{n-1} & a_{n-2} \\ a_{n-2} & -b_{n-2} & c_{n-2} & -d_{n-2} & a_{n-3} \\ a_{n-3} & -b_{n-3} & c_{n-3} & -d_{n-3} & a_{n-4} \\ a_{n-4} & -b_{n-4} & c_{n-4} & -d_{n-4} & a_{n-5} \end{bmatrix}$$

where  $a_n = \left\{ \begin{smallmatrix} n+4 \\ 4 \end{smallmatrix} \right\}_4$ ,  $b_n = \left\{ \begin{smallmatrix} n+4 \\ 3 \end{smallmatrix} \right\}_4 \left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\}_4$ ,  $c_n = \left\{ \begin{smallmatrix} n+4 \\ 2 \end{smallmatrix} \right\}_4 \left\{ \begin{smallmatrix} n+1 \\ 2 \end{smallmatrix} \right\}_4$ ,  $d_n = \left\{ \begin{smallmatrix} n+4 \\ 1 \end{smallmatrix} \right\}_4 \left\{ \begin{smallmatrix} n+2 \\ 3 \end{smallmatrix} \right\}_4$ .

The characteristic polynomial and the roots of  $G_{4,4}$  are given by

$$f_{4,4}(x) = \sum_{i=0}^5 (-1)^i \left\{ \begin{smallmatrix} 5 \\ i \end{smallmatrix} \right\}_4 x^{5-i}$$

and  $\lambda_5 = \alpha^{16}$ ,  $\lambda_4 = \beta^{16}$ ,  $\lambda_3 = \alpha^8$ ,  $\lambda_2 = \beta^8$ ,  $\lambda_1 = 1$  where  $\alpha, \beta = \left(p \pm \sqrt{p^2 + 4}\right) / 2$ , and so, the matrix  $H_5(v_4)$  has the same eigenvalues as the matrix  $G_{4,4}$ .

If we take  $k = 4$  and  $r = 3$ , then

$$G_{3,4} = \begin{bmatrix} a_1 & b_1 & -c_1 & -d_1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } H_{n,3,4} = \begin{bmatrix} a_n & b_n & -c_n & -d_n & a_{n-1} \\ a_{n-1} & b_{n-1} & -c_{n-1} & -d_{n-1} & a_{n-2} \\ a_{n-2} & b_{n-2} & -c_{n-2} & -d_{n-2} & a_{n-3} \\ a_{n-3} & b_{n-3} & -c_{n-3} & -d_{n-3} & a_{n-4} \\ a_{n-4} & b_{n-4} & -c_{n-4} & -d_{n-4} & a_{n-5} \end{bmatrix}$$



where  $a_n = \left\{ \begin{smallmatrix} n+4 \\ 4 \end{smallmatrix} \right\}_3$ ,  $b_n = \left\{ \begin{smallmatrix} n+4 \\ 3 \end{smallmatrix} \right\}_3 \left\{ \begin{smallmatrix} n \\ 1 \end{smallmatrix} \right\}_3$ ,  $c_n = \left\{ \begin{smallmatrix} n+4 \\ 2 \end{smallmatrix} \right\}_3 \left\{ \begin{smallmatrix} n+1 \\ 2 \end{smallmatrix} \right\}_3$ ,  $d_n = \left\{ \begin{smallmatrix} n+4 \\ 1 \end{smallmatrix} \right\}_3 \left\{ \begin{smallmatrix} n+2 \\ 3 \end{smallmatrix} \right\}_3$ . The characteristic polynomial and the roots of  $G_{3,4}$  are given by

$$f_{3,4}(x) = \sum_{i=0}^5 (-1)^{i(i+1)/2} \left\{ \begin{smallmatrix} 5 \\ i \end{smallmatrix} \right\}_3 x^{5-i}$$

and  $\lambda_5 = \alpha^{12}$ ,  $\lambda_4 = \beta^{12}$ ,  $\lambda_3 = -\alpha^6$ ,  $\lambda_2 = -\beta^6$ ,  $\lambda_1 = 1$ .

Generalizing this discussion, we can give the following result.

**Corollary 8.** *For  $n, k, r > 0$ ,*

$$\prod_{i=1}^{k+1} (x - \lambda_i) = \sum_{t=0}^{k+1} (-s(t-1)) \left\{ \begin{smallmatrix} k+1 \\ t \end{smallmatrix} \right\}_r x^{k+1-t}.$$

In [4], Cooper and Kennedy show that

$$\text{tr}(P_k(v_1)) = \frac{u_{kn}}{u_n},$$

where  $H_n(p)$  is the generalized Pascal matrix. This was extended in one direction by us (along with G.N. Stănică) in [17] to show

$$\text{tr}(P_k^n(v_r)) = \frac{u_{rkn}}{u_k}.$$

Since the matrices  $H_{n,r,k}$  and  $P_{k-1}^n(v_r)$  have the same eigenvalues, we also get

$$\text{tr}(H_{n,r,k-1}) = \frac{u_{rkn}}{u_r},$$

which easily implies the next result.

**Theorem 9.** *For  $n > 0$ ,*

$$\text{tr}(H_{n,r,k}) = \sum_{i=0}^{\lfloor k-1/2 \rfloor} (-1)^{inr} v_{(k-2i)nr} + \frac{1}{2} \left( 1 + (-1)^k \right).$$

From Proposition 7, we explicitly find the eigenvalues of the generalized  $k : C$ -nomial matrix.

## 4 The diagonalization of $G_{r,k}$ and the generalized Binet formula

In this section we diagonalize the matrix  $G_{r,k}$  and then derive the Binet formula for the  $r : C$ -nomial coefficients  $\left\{ \begin{smallmatrix} k \\ m \end{smallmatrix} \right\}_r$ . Let  $\lambda_1, \lambda_2, \dots, \lambda_{k+1}$  be the eigenvalues of  $G_{r,k}$ . Since the eigenvalues are all distinct, we can diagonalize  $G_{r,k}$ .

We define the  $(k+1) \times (k+1)$  Vandermonde matrix  $V$  and diagonal matrix  $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{k+1})$  as shown:

$$V = \begin{bmatrix} \lambda_1^k & \lambda_2^k & \dots & \lambda_{k+1}^k \\ \vdots & \vdots & & \vdots \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_{k+1}^2 \\ \lambda_1 & \lambda_2 & \dots & \lambda_{k+1} \\ 1 & 1 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_{k+1} \end{bmatrix}.$$

Since  $\lambda_i \neq \lambda_j$  for  $1 \leq i, j \leq k+1$ ,  $\det V \neq 0$ .

Let  $V_j^{(i)}$  is the  $(k+1) \times (k+1)$  matrix obtained from the transpose  $V^T$  by replacing the  $j$ th column of  $V$  by  $w_i$  where

$$w_i = [\lambda_1^{n-i+k+1} \quad \lambda_2^{n-i+k+1} \quad \dots \quad \lambda_{k+1}^{n-i+k+1}]^T.$$

Now we give the Binet formula for the generalized  $r : C$ -nomial coefficients. The case  $r = 1$  is given in [14]. The cases  $r > 1$  can be easily obtained similar to the case  $r = 1$ .

**Theorem 10.** For  $n, r, k > 0$ , and  $a_{r,s}$  as in (6), then

$$a_{n-i+1,j} = \frac{\det(V_j^{(i)})}{\det(V)}.$$

Let  $V_j^{(e_i)}$  be a  $(k+1) \times (k+1)$  matrix obtained from the Vandermonde matrix  $V$  by replacing the  $j$ th column of  $V$  by  $e_i$  where  $V$  is defined as before and  $e_i$  is the  $i$ th element of the natural basis for  $\mathbb{R}^n$ , that is,

$$V_j^{(e_i)} = \begin{bmatrix} \lambda_1^k & \dots & \lambda_{j-1}^k & 0 & \lambda_{j+1}^k & \dots & \lambda_{k+1}^k \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1^{k-i+1} & \dots & \lambda_{j-1}^{k-i+1} & 0 & \lambda_{j+1}^{k-i+1} & \dots & \lambda_{k+1}^{k-i+1} \\ \lambda_1^{k-i} & \dots & \lambda_{j-1}^{k-i} & 1 & \lambda_{j+1}^{k-i} & \dots & \lambda_{k+1}^{k-i} \\ \lambda_1^{k-i-1} & \dots & \lambda_{j-1}^{k-i-1} & 0 & \lambda_{j+1}^{k-i-1} & \dots & \lambda_{k+1}^{k-i-1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \lambda_1 & \dots & \lambda_{j-1} & 0 & \lambda_{j+1} & \dots & \lambda_{k+1} \\ 1 & \dots & 1 & 0 & 1 & \dots & 1 \end{bmatrix}.$$

$\downarrow$   
 $e_i$

Let  $q_j^{(i)} = \frac{|V_j^{(e_i)}|}{|V|}$  where the  $(k+1) \times (k+1)$  matrices  $V_j^{(e_i)}$  and  $V$  are defined as before.

We give the following theorem, whose proof is straightforward (the case  $r = 1$  is given in [14], and the remaining cases,  $r > 1$  can be similarly obtained).

**Theorem 11.** Let  $\lambda_1, \lambda_2, \dots, \lambda_{k+1}$  be the distinct roots of  $x^{k+1} - a_{1,1}x^k - a_{1,2}x^{k-1} - \dots - a_{1,k}x - a_{1,k+1} = 0$ . For any integer  $n$  and  $1 \leq i \leq k+1$ ,

$$a_{n,i} = \sum_{j=1}^{k+1} q_j^{(i)} \lambda_j^{n+k}.$$

For example, when  $A = 1$ , we get  $\alpha, \beta = (1 \pm \sqrt{5})/2$ . For  $k = 2, r = 3$  and  $A = 1$ , the roots of  $x^3 - \frac{F_9}{F_3}x^2 - \frac{F_9}{F_3}x + 1 = 0$  are  $\gamma_1 = \alpha^6, \gamma_2 = \beta^6, \gamma_3 = -1$ . By some computations, we get

$$\begin{aligned} q_1^{(1)} &= \frac{1}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)}, \quad q_2^{(1)} = \frac{1}{(\gamma_2 - \gamma_3)(\gamma_2 - \gamma_1)}, \quad q_3^{(1)} = \frac{1}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)}, \\ q_1^{(2)} &= -\frac{\gamma_2 + \gamma_3}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)}, \quad q_2^{(2)} = \frac{\gamma_1 + \gamma_3}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_2)}, \quad q_3^{(2)} = -\frac{\gamma_1 + \gamma_2}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)}, \\ q_1^{(3)} &= \frac{\gamma_2 \gamma_3}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)}, \quad q_2^{(3)} = -\frac{\gamma_1 \gamma_3}{(\gamma_1 - \gamma_2)(\gamma_2 - \gamma_3)}, \quad q_3^{(3)} = \frac{\gamma_1 \gamma_2}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)}. \end{aligned}$$

Thus, by Theorem 11 and some arrangements, we get

$$\begin{aligned} F_{3(n+1)}F_{3(n+2)} &= F_3F_6 \left( \frac{\gamma_1^{(n+2)}}{(\gamma_1 - \gamma_3)(\gamma_1 - \gamma_2)} + \frac{\gamma_2^{(n+2)}}{(\gamma_2 - \gamma_3)(\gamma_2 - \gamma_1)} + \frac{(-1)^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)} \right) \\ &= \frac{F_{6n+12} + F_{6n+6} + F_6(-1)^n}{10}, \end{aligned}$$

and

$$\begin{aligned} F_{3n}F_{3(n+2)} &= F_3F_3 \left( -\frac{(\gamma_2 + \gamma_3)\gamma_1^{n+2}}{(\gamma_1 - \gamma_2)(\gamma_1 - \gamma_3)} + \frac{(\gamma_1 + \gamma_3)\gamma_2^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_2)} - \frac{(\gamma_1 + \gamma_2)\gamma_3^{n+2}}{(\gamma_2 - \gamma_3)(\gamma_1 - \gamma_3)} \right) \\ &= \frac{F_{6(n+2)} - F_{6n} - F_{12}(-1)^n}{40}. \end{aligned}$$

## 5 Sums and a generating function for the generalized $C$ -nomial coefficients

In this section, we consider the sum of the generalized  $r : C$ -nomial coefficients by matrix methods. For this sum, we define a new matrix by extending  $G_{r,k}$  (8). Take the  $(k+2) \times (k+2)$  matrices

$$T_{r,k} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & & & \\ 0 & G_{r,k} & & \\ \vdots & & & \\ 0 & & & \end{bmatrix} \quad \text{and} \quad W_{n,r,k} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ S_{n,r} & & & \\ \vdots & & H_{n,r,k} & \\ S_{n-k,r} & & & \end{bmatrix}$$

where  $S_{n,r}$  is given by

$$S_{n,r} = \sum_{i=0}^{n-1} a_{i,1} = \sum_{i=0}^{n-1} \left\{ \begin{matrix} k+i \\ k \end{matrix} \right\}_r.$$

Then we have the following result.

**Theorem 12.** For  $n, r, k > 0$ ,

$$T_{r,k}^n = W_{n,r,k}.$$

*Proof.* Since  $S_{n+1,r} = a_{n,1} + S_{n,r}$  and by Theorem 2, we write the matrix recurrence relation  $W_{n,r,k} = W_{n-1,r,k}T_{r,k}$ . By the induction method, we write  $W_{n,k} = W_{1,k}T_k^{n-1}$ . From the definition of  $W_{n,k}$ , we obtain  $W_{1,r,k} = T_{r,k}^1$  and so  $W_{n,r,k} = T_{r,k}^n$ , and the proof is complete.  $\square$

From Proposition 7, we know that the polynomial  $f_{r,k}$  has root 1 for even  $r$  and  $k \equiv 0 \pmod{4}$ . Expanding  $\det(\lambda I_{k+2} - T_{r,k})$  with respect to the first row, it is easily seen that the matrix  $T_{r,k}$  has also the eigenvalue 1. Thus the matrix  $T_{r,k}$  has a double eigenvalue for even  $r$  and  $k \equiv 0 \pmod{4}$ . For odd  $r$  and  $k \not\equiv 0 \pmod{4}$ , we can diagonalize the matrix  $T_{r,k}$  and so we derive an explicit formula for the sum.

Define a  $(k+2) \times (k+2)$  matrix  $M$  as shown:

$$M = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \delta & & & \\ \vdots & & V & \\ \delta & & & \end{bmatrix}$$

where  $\delta = \left(1 - \sum_{i=1}^{k+1} a_{1,i}\right)^{-1}$  and the Vandermonde matrix  $V$  is defined as before.

It is easy to see that  $T_{r,k}M = MD_1$ , where  $D_1 = \text{diag}(1, \lambda_1, \dots, \lambda_{k+1})$ . By the Vandermonde matrix  $V$ , computing  $\det M$  with respect to the first row shows  $\det M = \det V$ .

**Theorem 13.** For  $n, k > 0$ ,  $k \not\equiv 0 \pmod{4}$  and odd  $r > 0$

$$S_{n,r} = \frac{a_{n,1} + a_{n,2} + \dots + a_{n,k+1} - 1}{\sum_{i=1}^{k+1} a_{1,i} - 1}.$$

*Proof.* Since  $M$  is invertible,  $M^{-1}T_{r,k}M = D_1$ , that is,  $T_{r,k}$  is similar to  $D_1$ . Thus we write  $T_k^n M = MD_1^n$ . By Theorem 12,  $W_{n,r,k}M = MD_1^n$ . Equating the  $(2, 1)$ th elements of  $W_{n,r,k}M = MD_1^n$  and from a matrix multiplication, the proof follows.  $\square$

From Theorem 13, for  $r = k = 3$  and  $n \geq 0$ , one may obtain

$$\sum_{i=0}^n \left\{ \begin{matrix} 3+i \\ 3 \end{matrix} \right\}_3 = \frac{\left\{ \begin{matrix} n+3 \\ 3 \end{matrix} \right\}_3 + \left\{ \begin{matrix} n+3 \\ 2 \end{matrix} \right\}_3 \left\{ \begin{matrix} n \\ 1 \end{matrix} \right\}_3 - \left\{ \begin{matrix} n+3 \\ 1 \end{matrix} \right\}_3 \left\{ \begin{matrix} n+1 \\ 2 \end{matrix} \right\}_3 - \left\{ \begin{matrix} n+2 \\ 3 \end{matrix} \right\}_3 - 1}{\left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\}_3 + \left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\}_3 - \left\{ \begin{matrix} 4 \\ 1 \end{matrix} \right\}_3 - \left\{ \begin{matrix} 4 \\ 0 \end{matrix} \right\}_3 - 1}.$$

Next, we display a generating function for the generalized  $C$ -nomial coefficients. Define

$$g(i, x) = a_{0,i} + a_{1,i}x + a_{2,i}x^2 + a_{3,i}x^3 + \dots + a_{n,i}x^n + \dots$$

where  $a_{n,i}$  is defined as before. The proof of the following proposition is rather straightforward and it is left as an exercise to the reader.

**Proposition 14.** For  $1 \leq i \leq k+1$  and  $1 \leq t \leq k$

$$g(i, x) = \frac{a_{0,i} + \left(a_{t,i} - \sum_{m=1}^t a_{1,m}a_{t-m,i}\right)x^t}{1 - a_{1,1}x - a_{1,2}x^2 - \dots - a_{1,k+1}x^{k+1}}.$$

For example, when  $i = 1$  in Proposition 14, we get, for odd  $r$ ,

$$\begin{aligned} & \sum_{n=0}^{\infty} \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\}_r x^n \\ &= \frac{1}{1 - \left\{ \begin{matrix} k+1 \\ k \end{matrix} \right\}_r x - \left\{ \begin{matrix} k+1 \\ k-1 \end{matrix} \right\}_r x^2 + \cdots + (-1)^{k(k+1)/2} \left\{ \begin{matrix} k+1 \\ 1 \end{matrix} \right\}_r x^k + (-1)^{(k+1)(k+2)/2} x^{k+1}} \end{aligned}$$

and for even  $r$

$$\begin{aligned} & \sum_{n=0}^{\infty} \left\{ \begin{matrix} n+k \\ k \end{matrix} \right\}_r x^n \\ &= \frac{1}{1 - \left\{ \begin{matrix} k+1 \\ k \end{matrix} \right\}_r x + \left\{ \begin{matrix} k+1 \\ k-1 \end{matrix} \right\}_r x^2 + \cdots + (-1)^k \left\{ \begin{matrix} k+1 \\ 1 \end{matrix} \right\}_r x^k + (-1)^{(k+1)} x^{k+1}}. \end{aligned}$$

For  $k = r = 3$  and  $i = 1$ , we get

$$\sum_{n=0}^{\infty} \left\{ \begin{matrix} n+3 \\ 3 \end{matrix} \right\}_3 x^n = \frac{1}{1 - \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\}_3 x - \left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\}_3 x^2 + \left\{ \begin{matrix} 4 \\ 1 \end{matrix} \right\}_3 x^3 + x^4}.$$

Similarly, one can obtain a plethora of other identities by taking other values of the involved parameters.

## References

- [1] P. Bachmann, *Niedere Zahlentheorie* (Zweiter Teil), Leipzig, Teubner, 1910; republished as *Additive Zahlentheorie* by Chelsea Publishing Co., New York, 1968.
- [2] L. Carlitz, The characteristic polynomial of a certain matrix of binomial coefficients, *Fibonacci Quart.* **3** (1965), 81–89.
- [3] R. D. Carmichael, On the numerical factors of the arithmetic forms  $\alpha^n \pm \beta^n$ , *Annals of Math.* **15** (1913–1914), 30–70.
- [4] C. Cooper and R. Kennedy, Proof of a result by Jarden by generalizing a proof by Carlitz, *Fibonacci Quart.* **33** (1995), 304–310.
- [5] P. Duvall and T. Vaughan, Pell polynomials and a conjecture of Mahon and Horadam, *Fibonacci Quart.* **26** (1988), 344–353.
- [6] G. Fontené, Généralisation d’une formule connue, *Nouv. Ann. Math.* **4** (1915), 112.
- [7] H.W. Gould, The bracket function and Fontené–Ward generalized binomial coefficients with application to Fibonomial coefficients, *Fibonacci Quart.* **7** (1969), 23–40.
- [8] A.P. Hillman and V.E. Hoggatt, The characteristic polynomial of the generalized shift matrix, *Fibonacci Quart.* **3** (1965), 91–94.

- [9] V.E. Hoggatt, Fibonacci numbers and generalized binomial coefficients, *Fibonacci Quart.* **5** (1967), 383–400.
- [10] V.E. Hoggatt and M. Bicknell, Fourth power Fibonacci identities from Pascal's triangle, *Fibonacci Quart.* **2** (1964), 81–89.
- [11] A.F. Horadam, Generating functions for powers of a certain generalized sequence of numbers, *Duke Math. J.* **32** (1965), 437–446.
- [12] D. Jarden, Recurring sequences, *Riveon Lematematika*, Jerusalem, Israel, 1958, 42–45.
- [13] D. Jarden, T. Motzkin, The product of sequences with a common linear recursion formula of order 2, *Riveon Lematematika* **3** (1949), 25–27, 38.
- [14] E. Kilic, The generalized Fibonomial matrix, *European J. Comb.* **31** (2010), 193–209.
- [15] E. Kilic, Sums of the squares of terms of sequence  $\{u_n\}$ , *Proc. Indian Acad. Sci. (Math. Sci.)* **118** (2008), 27–41.
- [16] E. Kilic, N. Omur and Y. Ulutas, Matrix representations for the second order recurrence  $\{U_{kn}\}$ , accepted in *Ars Combin.*
- [17] E. Kilic, G.N. Stănică and P. Stănică, Spectral properties of some combinatorial matrices, *Congressus Numerantium Vol. 201* (2010), Proc. Internat. Conf. Fibonacci Numbers & Applic. – '08, 223–236.
- [18] E. Kilic and P. Stănică, The Lehmer matrix and its recursive analogue, accepted in *J. of Combinat. Math. and Combinat. Computing*.
- [19] F. Luca, D. Marques, P. Stănică, On the spacings of C-nomial coefficients, *J. Number Theory* **130** (2010), 82–100.
- [20] J. Seibert and P. Trojovsky, On some identities for the Fibonomial coefficients, *Math. Slovaca* **55** (2005), 9–19.
- [21] P. Stănică, Netted matrices, *Int. J. Math. Math. Sci.* **39** (2003), 2507–2518.
- [22] R. Torretto, A. Fuchs, Generalized Binomial coefficients, *Fibonacci Quart.* **2** (1964), 296–302.
- [23] P. Trojovsky, On some identities for the Fibonomial coefficients via generating function, *Discrete Appl. Math.* **155** (2007), 2017–2024.
- [24] S. Vajda, *Fibonacci & Lucas numbers, and the golden section*, John Wiley & Sons, Inc., New York, 1989.
- [25] M. Ward, A calculus of sequences, *Amer. J. Math.* **58** (1936), 255–266.